

XXIII. *The Principles and Illustration of an advantageous Method of arranging the Differences of Logarithms, on lines graduated for the Purpose of Computation.* By Mr. William Nicholson; communicated by Sir Joseph Banks, Bart. P. R. S.

Read March 29, 1787.

1. IF two geometrical series of numbers, having the same common ratio, be placed in order with the terms opposite each other; the ratio, between any term in one series and its opposite in the other, will be constant *.

2. And likewise the ratio of a term in one series to any term in the other, will be the same as obtains between any other two terms having the same relative position and distance †.

3. In all such pairs of geometrical series, as have the same common ratio, the last mentioned property obtains, though the first antecedent and consequent be taken in one pair, and the second in any other pair ‡.

$$* \text{ Geom. series } \begin{cases} a & an & an^2 & an^3 & an^4 \\ b & bn & bn^2 & bn^3 & bn^4 \end{cases}$$

$$\text{Then } a : b :: an : bn :: an^2 : bn^2, \text{ \&c.}$$

$$† \text{ In the foregoing series } a : bn^2 :: an^2 : bn^4 :: an : bn^3, \text{ \&c.}$$

$$‡ \text{ Geo. series } \begin{cases} a & an & an^2 & an^3 & an^4 \\ b & bn & bn^2 & bn^3 & bn^4 \end{cases}$$

$$\text{Geo. series } \begin{cases} d & dn & dn^2 & dn^3 & dn^4 \\ \frac{bd}{a} & \frac{bdn}{a} & \frac{bdn^2}{a} & \frac{bdn^3}{a} & \frac{bdn^4}{a} \end{cases}$$

$$\text{Then } a : bn^2 :: dn : \frac{bdn^3}{a}, \text{ \&c.}$$

4. If

4. If the differences of the logarithms of numbers be laid in order upon an arrangement of equi-distant parallel right lines, in such a manner as that a right line, drawn across the whole, shall intersect it at divisions which denote numbers in geometrical progression; then, from the condition of the arrangement and the property of this logarithmic line, it follows, first, that every right line, so drawn, will, by its intersections, indicate a geometrical series of numbers*; secondly,

* Let AB, CD, EF (Tab. X. fig. 5.) be portions of the logarithmic line, arranged according to the condition: let GH be a right line drawn across, so as to pass through points of division e, c, a , denoting numbers in geometrical progression: then will any other line IK, drawn across the arrangement, also pass through points f, d, b , denoting numbers in geometrical progression.

Demonstration. From one of the extreme points of intersection f in the last named line IK draw the right line fg , parallel to GH, and intersecting the arrangement in the points i, b ; and the ratios of the numbers $e : f$, $c : i$, and $a : b$, will be equal, because the intervals on the logarithmic line, or differences of the logarithms of those numbers, are equal:

$$\text{Or } \frac{e}{c} = \frac{f}{i} \text{ and } \frac{c}{a} = \frac{i}{b}.$$

$$\text{But } \frac{e}{c} = \frac{c}{a} \text{ by the condition.}$$

Therefore $\frac{f}{i} = \frac{i}{b}$; or the numbers f, i, b , are in the same continued ratio as the numbers e, c, a .

Again, the point f , the line id , and the line bb , are in arithmetical progression, and denote the differences of the logarithms of the numbers f and f , i and d , b and b .

The quotients of the numbers themselves are therefore in geometrical progression, that is,

$$\frac{f}{f} : \frac{d}{i} : \frac{b}{b}, \text{ or } \frac{i}{d} = \frac{db}{bi}.$$

$$\text{Or } \frac{i}{d} = \frac{di}{bf}, \text{ by substituting } \frac{i}{f} \text{ for its equal } \frac{b}{i}.$$

$$\text{Whence } \frac{f}{d} = \frac{d}{b} \text{ or } f : d : b. \quad \text{Q. E. D.}$$

that such series, as are so indicated by parallel right lines, will have the same common ratio*; and, thirdly, that the series thus indicated by two parallel right lines, supposed to move laterally without changing either their mutual distance, or parallelism to themselves, will have each the same common ratio; and in all pairs of series indicated by such two lines, the ratio between an antecedent on one parallel and the opposite term on the other, taken as a consequent, will be constant†.

5. Thus far the logarithmic line has been considered as unlimited. If, therefore, an antecedent and consequent be given, it will be possible to find both on the arrangement, and to draw two parallel lines, one over each number: and if the lines be then supposed to move, without changing either their distance or absolute direction, so that the line, which before marked an antecedent, may now mark a new antecedent; the other (by 2. and 3.) will mark a number, at the same relative position and distance, which shall be the consequent to this last antecedent after the same ratio.

6. Suppose a logarithmic line to contain no more than a single range of numbers from 1 to 10, it will not be necessary, for the purposes of computation, to repeat it; for if a slider

* In the same manner as it was proved that the line fg , parallel to GH , passes through points of division, denoting numbers in the same continued ratio as those indicated by the line GH , it may also be shewn, that the line LM parallel to any other line IK , will pass through a series of numeral points, having the same continued ratio as the series indicated by that line IK to which it is parallel.

† Because the lines preserve their parallelism to their former situation, they will indicate geometrical series having the same common ratio as before; and, because their distance measured on the logarithmic line remains unchanged, the differences of the logarithms of opposite numbers, and consequently their ratio, will be constant.

or beam have two fixed points at the distance of the interval between 1 and 10, and a moveable point be made to range between these (always to indicate the antecedent); then, I say, if the consequent fixed point fall without the rule, the other fixed point will shew the division it would have fallen on, if the rule had been prolonged. This may be easily applied to the arrangement described, N^o 4.

7. If the arrangement consist only of the logarithms from 1 to 10, and the parallel cross lines intersect that geometrical series whose successive ratios altogether, with that of the last to the first, make by composition the ratio $\frac{1}{10}$, the contrivance, N^o 6, may be applied to shew such consequents as fall, laterally, without the rule.

8. It is convenient that the arrangement of the lines be disposed so as to occupy a rectangular parallelogram; or, in other words, that the cross line, cutting the series last mentioned, may be at right angles to the length of the rule.

The construction of an instrument on the foregoing principles admits of various dispositions of the graduated lines and apparatus for measuring intervals upon them. Fig. 1. is a rule consisting of ten parallel lines, equivalent to a double line of numbers upwards of 20 feet in length. Fig. 2. is a beam compass for measuring intervals. The parts B, A, C, apply to the surface of the rule; the middle A being moveable sideways in a groove in the piece DE, so as always to preserve its parallelism to the external pieces B, C, which are fixed at a distance equal to the length of the rule, and have their edges placed according to the condition in § 7. which is here at right angles to the length. The piece DE, in the use, is applied to the edge FG of the rule. The edges or borders H, I, K, L, may

may be made of transparent horn or tortoise-shell, which is in many respects more convenient than any opaque substance.

The Use. Apply the edge of either B or C to the consequent and slide the piece A to the antecedent, observing the difference between the numbers on the pieces denoting the lines they are found on; then, if the same edge of A be applied to any other antecedent, the other piece B or C, made use of, will intersect a consequent in the same ratio upon that line of the arrangement, which has the same situation, with respect to the antecedent, as the line of the former consequent had to its antecedent. The numbers on the pieces serve to indicate the relative situations. But if B be the consequent piece, and fall without the rule, the piece C will shew the consequent one line lower; or if C, in the like case, fall without the rule, B will shew the consequent one line higher. It would be easy to make the same kind of provision for the numbers which fall lateral without the rule: and it might be found convenient if, for the purpose of computation, instruments of this kind were to be made with an hundred or more lines. But in the present instrument, the numbers on the pieces will answer the former purpose: for if a consequent fall on a line at any given number of intervals without the rule, it will be found on that line of the arrangement, which occupies the same number of intervals, reckoned inwards from the opposite edge of the rule.

Fig. 3. is a GUNTER's scale, equivalent to that of 28 inches in length, published by the late Mr. ROBERTSON. It is, however, but one-fourth of the length, and contains only one-fourth of the quantity of division. In the slider GH is a moveable piece AB, across which a fine line is drawn; and there are also lines CD, EF, drawn across the slider, at a distance from each other equal to the length of the rule. The use of this is similar to that of the foregoing. The

line CD or EF is to be placed at the consequent, and the line in the piece AB at the antecedent: then, if the piece AB be placed at any other antecedent, the same line CD or EF will indicate its consequent in the same ratio taken the same way; that is to say, if the antecedent and the consequent lie on the same side of the slider, all other antecedents and consequents in that ratio will lie in the same manner; and the contrary if they do not, &c. But if the consequent line fall without the rule, the other fixed line on the slider will shew the consequent; but on the contrary side of the slider to that where it would else have been seen by means of the first consequent line.

Fig. 4. is an instrument equivalent to the same rule of $28\frac{1}{2}$ inches long. It consists of three concentric circles engraved and graduated on a plate of about $1\frac{1}{2}$ inch in diameter. From the center proceed two legs A, B, having right-lined edges in the direction of radii. They are moveable either singly or together. To use this instrument, place one of the edges at the antecedent, and the other at the consequent, and fix them to that angle. The two legs being then moved together, and the antecedent leg placed at any other number, the other leg will give its consequent in the like position or situation on the lines. If the line CD happen to lie between the legs, and B be the consequent leg, the number sought will be found one line farther from the center than it would otherwise have been; and, on the contrary, it will be found one line nearer in the like case, if A be the consequent leg. This instrument, differing from fig. 1. only in its circular form, and the advantages resulting from that form, the lines must be taken to succeed each other in the same manner laterally; so that numbers, which fall either without or within the arrangement of circles, will

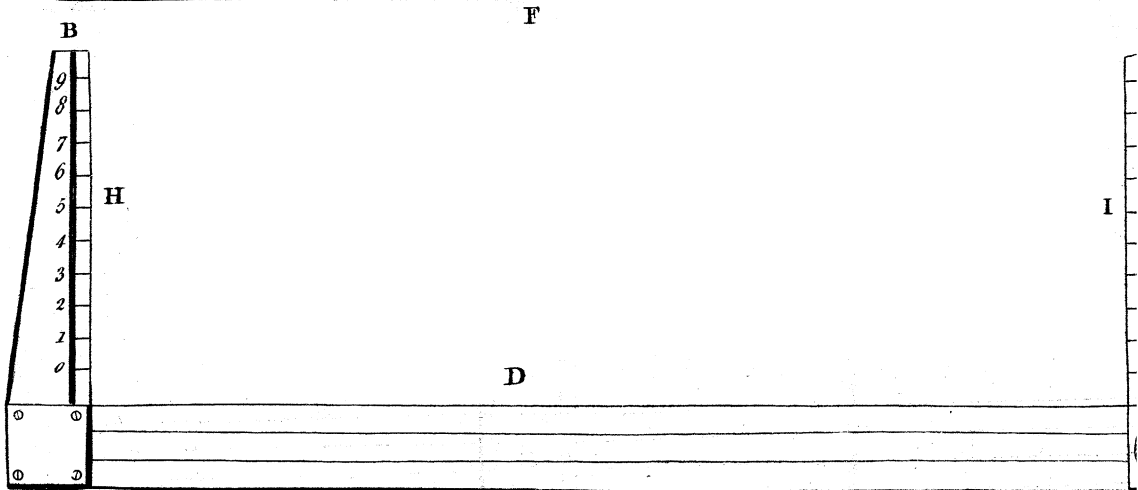
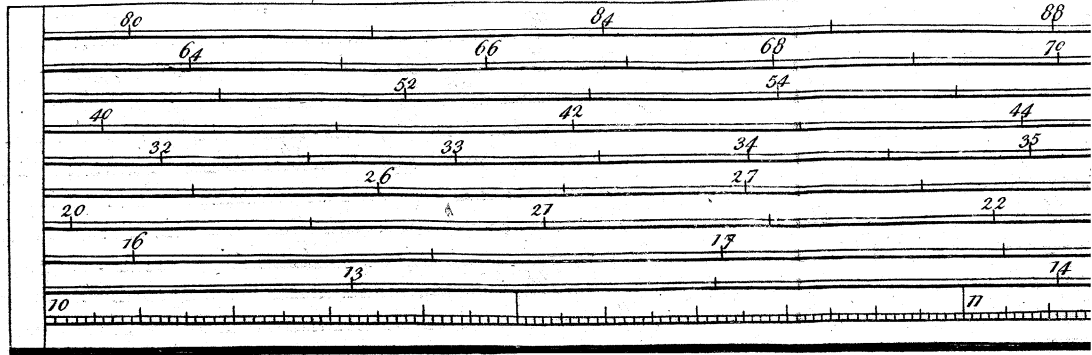
be found on such lines of the arrangement as would have occupied the vacant places if the succession of lines had been indefinitely repeated sideways.

I approve of this construction, as superior to every other which has yet occurred to me, not only in point of convenience, but likewise in the probability of being better executed, because small arcs may be graduated with very great accuracy, by divisions transferred from a larger original. The instrument, fig. 1. may be conveniently contained in a circle of about $4\frac{1}{2}$ inches diameter.

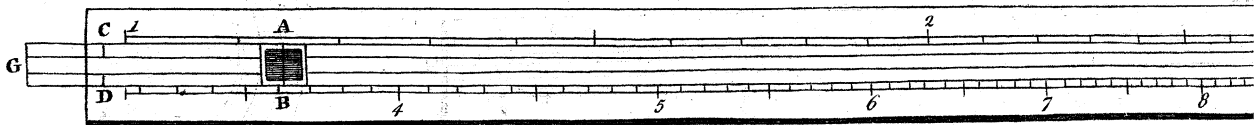
The circular instrument is a combination of the GUNTER's line and the sector, with the improvements here pointed out. The property of the sector may be useful in magnifying the differences of the logarithms in the upper part of the line of fines, the middle of the tangents, or the beginning of the versed fines. It is even possible, as mathematicians will easily conceive, to draw spirals on which graduations of parts, every where equal to each other, will shew the ratios of those lines by means of moveable radii similar to those in this instrument. But I do not, in this Discourse, propose to enter into enquiries respecting the nature of such curves, nor their utility in the present business.



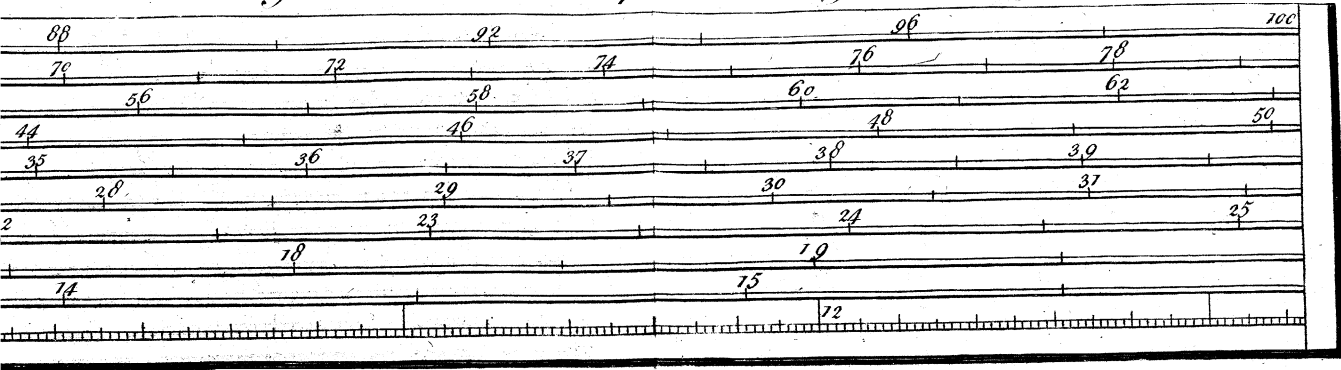
Sketch of an Instrument, equivalent to a c



Gunter's Scale, equivalent to those commonly made of 28 1/2 Inches to



to a common Gunter's Scale 20 feet in length. Fig. 1.



G

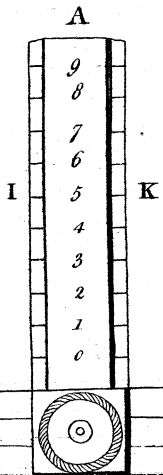
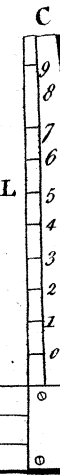
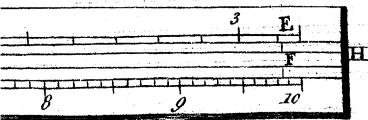


Fig. 2.

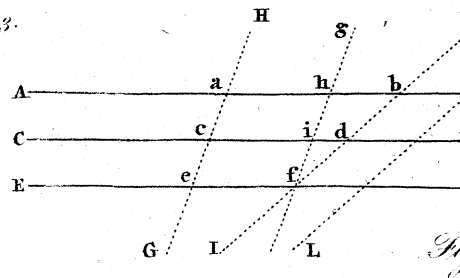
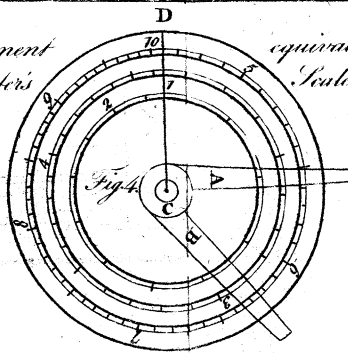
E



ches long. Fig. 3.



Instrument to the Gunter's equivalent to the Gunter's Scale, Fig. 3.



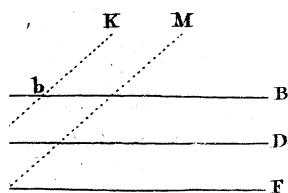
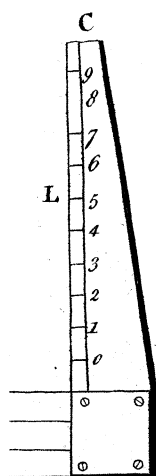
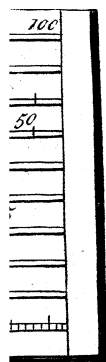


Fig. 5.

Sketch of an Instrument, equivalent to a common Gunter's Scale 20 feet in length. Fig. 1.

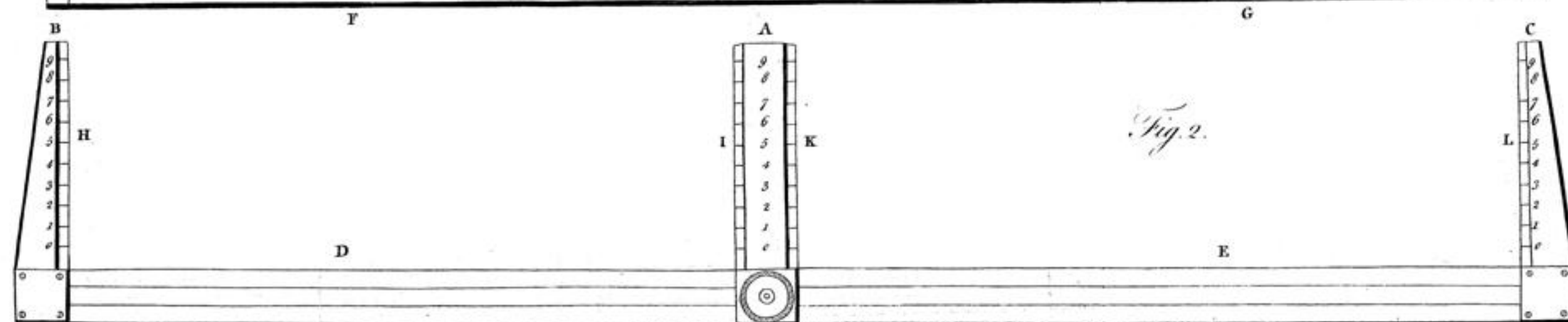
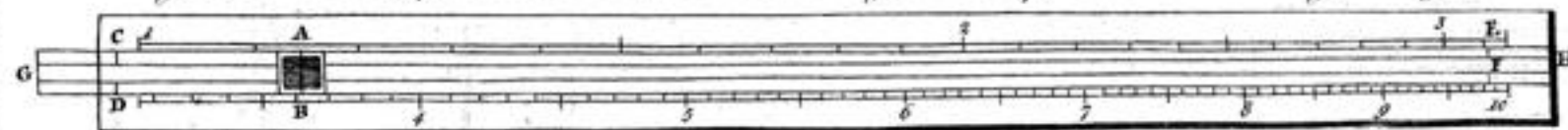


Fig. 2.

Gunter's Scale, equivalent to those commonly made of 28 1/2 Inches long. Fig. 3.



Instrument equivalent to the Gunter's Scale, Fig. 3.

